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COUNTABLE MODELS OF SMALL DEPENDENT THEORIES

ABSTRACT

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Actuality of the research theme. At the present time, one the main problems of model theory include the solution of the spectral problem, that is, a description for the different classes of theories of the function $I(T,\lambda)$, the number of non-isomorphic models of the theory T of the cardinality λ.

Among insufficiently explored problems of model theory still remains the problem of the description of the number $I(T,\omega)$ of countable non-isomorphic models of the theory T.

Related with this issue is the Vaught's conjecture, or the Vaught's hypothesis, according to which there is no countable theory for which the number of countable non-isomorphic models is greater than the cardinality of natural numbers and less than the cardinality of real numbers, that is, there is no theory satisfying the condition $\omega < I(T,\omega) < 2^{\omega}$.

Morley proved that if is infinite then it must be ω or 2^{ω} or the cardinality of between ω and the cardinality of continuum. That is, $I(T,\omega) \in \omega \cup \{\omega, \omega_1, 2^{\omega}\}$. Vaught proved that the number of countable models cannot be equal to two.

A theory is said to be small, if the number of all its n-types over an empty set is no more than countable for every finite number n. In the case when a given theory is not small, then number of countable nonisomorphic models of it is maximal, that is equal to 2^{ω}.

J. Baldwin and A. Lachlan confirmed the Vaught conjecture for the class of uncountably categorical theories. For the class of omega-stable theories the conjecture was confirmed by MacKay, Harrington and Shelah. L. Mayer by using of a theory of orthogonality of 1-types in ordered minimal theories of D. Marker proved the Vaught conjecture for the class of ordered minimal theories. The Vaught conjecture for quite o-minimal theories was confirmed by S.V. Sudoplatov and B.Sh. Kulpeshov.

Although the Vaught conjecture was confirmed for individual classes of complete theories, in general case the task of counting the number of countable models is still not solved. One of the classes for which the Vaught conjecture has not been proven yet is the class of dependent theories. Exactly this class is under investigation of this research.

The description of cases, in which a complete countable theories have the maximal, that is 2^{ω}, number of countable non-isomorphic models, is an important question in studying the countable spectrum of those theories.

For instance, at first, L. Mayer found sufficient conditions for an o-minimal theory to have the maximal number of countable non-isomorphic model; and only after that she moved to proving the Vaught conjecture for o-minimal theories.

Another example is the work “Vaught’s conjecture for quite o-minimal theories” by S. Sudoplatov and B. Kulpeshov, in which the authors indicated the conditions of maximality of the countable spectrum, and proved the Vaught conjecture for quite o-minimal theories.

In this connection, a large part of the dissertation will be devoted to finding conditions under which a countable theory has the maximal number of countable non-isomorphic models.

The question on the number of countable models is described in the works of many scientists. The other works on this subject that are referenced by many authors are written by S. Shelah, A. Pillay and M. Benda. One more work is the article of S.V. Sudoplatov and R.A. Popkov “Distributions of countable models of theories with continuum many types” which classifies the theories which have the continuum number of types (and therefore the maximal number of models) according to different criteria. In his work “The number of countable models” Enrique Casanovas considered the number of countable non-isomorphic models from the different sides of view: semi-isolation, Rudin-Keisler order, smooth classes and closures, predimensions, dimension and stability. B.S. Baizhanov and B. Omarov considered the number of countable nonisomorphic models from the point of view of the notion of finite diagrams. At the present time there is no answer on the Vaught conjecture but model theory specialists continue to work on it, in particular S.V. Sudoplatov jointly with B.S. Baizhanov and V.V. Verbovskiy.

The number of countable models of theories with an $\emptyset$-definable relation of a linear order had been studied in the works of L. Mayer, M. Rubin, S. Shelah, B.S. Baizhanov, S.V. Sudoplatov and B.Sh. Kulpeshov and others. The question on the number of countable models of theories with linear and partial order has a big place in the dissertation, since it is of a big importance in the class of dependent, non-stable theories.

The aims and objectives of the study.

The work is devoted to studying countable spectrum of theories which have a countable number of types. The aims of the work are the following:

1. To find conditions of maximality of a number of countable models.

2. To find a class of dependent theories for which the Vaught conjecture can be solved.

The main provisions for the defence of the dissertation:

1. Given a countable complete theory of (an expansion of) a linear order. If there exists a finite subset of some model of this theory and a non-principal extremely trivial 1-type over this subset, then the theory has 2^{ω} countable non-isomorphic models.

2. If there exists a formula determining a partial order on tuples such that there exists a finite discrete chain of length greater or equal to any given natural number, then the given a countable complete theory has the maximal number of countable non-isomorphic models.

3. If in a countable complete theory of (an expansion of) a linear order there exists a formula quasi-successor on some non-principal 1-type. Then this theory has 2^{ω} countable non-isomorphic models.

4. The class of weakly o-minimal theories of convexity rank 1 satisfy the Vaught conjecture.

The objects of research are small dependent theories.

The research subjects are countable models of small dependent theories and their number up to an isomorphism.

Research methods include analysis of theories through the use of properties of types. Neighbourhoods in a realization of a type are considered, that is how formulas behave inside the realization set of a given type, as well as relations of orthogonality between few types are considered: the weak and almost orthogonality between types allow us to understand in which way realizations of these types in models are connected. For example, realization of one type in a model can imply realization of one or more types in the same model, or all realizations of few types can be independent from each other, allowing all possible combinations of realizing-omitting these types in models of the theory.

Also, while constructing models, a method based on the Tarski-Vaught criterion is used. The Tarski-Vaught criterion guarantees for a subset of a model that it would be a model of the given theory (and moreover, it would be an elementary submodel of this model).

Scientific novelty of the dissertation research. Problem of description of a countable spectrum of small dependent theories is open at the present time. Classes of theories under the study have not been investigated on a number of countable models.

Theoretical and practical significance of the research. Researches in this area constitute steps in solving the Vaught conjecture. The obtained results on the nature of countable models of small countable theories can be applied to group, ring and field theory.

Connection of the dissertation thesis with the other scientific research works. The dissertation thesis was implemented within the scientific projects of the program of grant financing of fundamental researches in the area of natural sciences of the Ministry of education and science of the Republic of Kazakhstan “Properties of types in dependent theories” (2015-2017 years, 5125/GF4) and “Conservative extensions, countable ordered models and closure operators” (2018-2020 years, AP05134992).

The work approbation. Results of the work were presented and discussed at the following conferences and seminars:

Logic Colloquium 2015, University of Helsinki, Finland, 2015;

“Function theory, informatics, differential equations and their applications”, Almaty, 2015;

“Algebra, analysis, differential equations and their applications”, Almaty, 2016;

Logic Colloquium 2016, Leeds, United Kingdom, 2016;

Annual April scientific conference of the Institute of Mathematics and Mathematical Modeling, Almaty, 2017;

The International Summer School–Conference named “Problems Allied to Universal Algebra and Model Theory”, Erlagol – 2017, Novosibirsk, Russian Federation, 2017;

“Actual problems of pure and applied mathematics”, Almaty, 2017;

Scientific seminars of the department of algebra and mathematical logic of the Institute of Mathematics and mathematical modeling;

Results of the dissertation work were discussed with prominent model theory specialists during the scientific training at the University of Illinois at Chicago, USA, and were presented at the Louise Hay Logic Seminar of the university in November 2017.

Assessment of the completeness of the aims of the work. All the results are new and are based on our own methods and tools. Conditions, guaranteeing maximality of the number of countable models were obtained, as well as a subclass of dependent theories satisfying the Vaught conjecture was found. Therefore, the work’s objectives were fully completed.

Suggestions on applications of the obtained results. The results obtained in this area of model theory can be used during the study of countable models of small countable theories and during a search of a proof for the Vaught conjecture. For example, the conditions obtained for maximality of the number of countable models imply that a theory having ω_1 countable models should not satisfy those conditions. Results on the nature of countable nonisomorphic models of small theories can be applied theories of algebraic structures.

Assessment of scientific level of the work in comparison with the achievements in the scientific direction. The results obtained in comparison with the best achievements of foreign colleagues do not lose and contribute to the study of countable spectrum of small theories.

Publications. Based on the results of the dissertation 15 works were published: 5 journal articles (2 works in indexed journals, and 3 in journals recommended by the Committee for Control in Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan, and 10 works in proceedings of international scientific conferences.

Volume and structure of the dissertation. The work includes the title page, contents, normative references, definitions, notations and abbreviations, introduction, 8 sections, conclusion and references. Total volume of the dissertation is 79 pages, the work contains 1 illustration and 79 literature references.

Main content of the dissertation work. The introduction of the thesis includes actuality of the research theme, aims of the research, the main provisions for the defence of the dissertation, the research object and subject, methods, the novelty and theoretical and practical significance of the dissertation thesis, connection of the dissertation thesis with the other scientific research works, the work approbation,

author's publications, as well as volume, structure and content of the dissertation thesis.

The first section explains the current state of the investigated area of model theory.

The second section of the dissertation gives the reader preliminary information and explains basic tools which will be used throughout the dissertation, such as Tarski-Vaught criterion, omitting types and compactness.

The third section of the work considers the notion of dowries (in other words, finite diagrams), meaning, sets of all the types realized in a given model; and, under a given assumption considers the case of a counterexample of the Vaught conjecture, in this section it was shown that under the assumption that if there is a theory which has ω_1 countable non-isomorphic models, then there is a dowry which has ω_1 countable non-isomorphic models.

In the fourth section of the dissertation the notions of weak and almost orthogonality are introduced, some useful properties of types are proved, as well as few theorems connecting orthogonality with the number of countable models have been proved.

The 5th section is focused on finding conditions on small countable theories of a definable linear order to have the maximal number of countable non-isomorphic models. We introduce different notions of triviality of non-principal types, namely, the notions of an extremely trivial type, almost extremely trivial, and an eventually extremely trivial type; give examples; and explain connections between triviality of types and type-preserving formulas. We also prove that a theory of (an expansion) linear order, which has an extremely trivial type, has the maximal number of countable non-isomorphic models.

In Section six small countable theories with a definable partial order on tuples are studied, and a theorem on a sufficient condition of maximality of number of countable models of such theories is proved. This condition being existence for any given natural number of a discrete φ-chain of length greater or equal to that natural number.

The Section 7 studies countable theories with a definable linear order and introduces the notion of a so-called quasi-successor formula. Such a formula allows us to construct definable neighbourhoods inside a type. By using an approach introduced in the previous section, it can be proven that a theory of (an expansion of) a definable linear order which has the quasi-successor formula on a non-principal type over a finite subset of some model of the theory has the maximal number of countable non-isomorphic models.

In Section 8 we consider a subclass of dependent theories, the class of weakly o-minimal theories of a convexity rank 1. The binarity of such theories is proven and, by use of binarity, it was that such theories satisfy the Vaught conjecture. More precisely, it is proven that every weakly o-minimal theory of convexity rank 1 is either countably categorical, is an Ehrenfeucht theory, namely it has exactly k countable models, where $3 \leq k < \omega$; it has ω countable models; or has 2^{ω}, the maximal number of countable models.

To clarify the main result of the section we also show different examples of weakly o-minimal theories of the convexity rank 1 and count the number of countable models of those theories.

The conclusion lists and generalizes the main results obtained during the implementation of the dissertation thesis.